AN EXAMPLE OF A MINIMAL ACTION OF THE FREE SEMI-GROUP $\mathbb{F}_2^+$ ON THE HILBERT SPACE

by

Sophie Grivaux & Maria Roginskaya

Abstract. — The Invariant Subset Problem on the Hilbert space is to know whether there exists a bounded linear operator T on a separable infinite-dimensional Hilbert space H such that the orbit $\{T^n x;\ n \geq 0\}$ of every non-zero vector $x \in H$ under the action of T is dense in H. We show that there exists a bounded linear operator T on a complex separable infinite-dimensional Hilbert space H and a unitary operator V on H, such that the following property holds true: for every non-zero vector $x \in H$, either x or Vx has a dense orbit under the action of T. As a consequence, we obtain in particular that there exists a minimal action of the free semi-group with two generators $\mathbb{F}_2^+$ on a complex separable infinite-dimensional Hilbert space H.

1. Introduction

Let H be a complex separable infinite-dimensional Hilbert space, and $T \in \mathcal{B}(H)$ a bounded linear operator on H. Our present work is motivated by the well-known Invariant Subspace and Subset Problems, which can be easily stated as follows: if $T \in \mathcal{B}(H)$, does there always exist a closed subspace M of H (resp. a closed subset F of H), non-trivial in the sense that it is distinct from $\{0\}$ and H, and which is invariant under the action of T? It is not difficult to see that an operator T on H has no non-trivial invariant closed subspace (resp. subset) if and only if for every non-zero vector $x \in H$ the linear span of the orbit $\mathrm{Orb}(x,T) = \{T^n x;\ n \geq 0\}$ of x under the action of T (resp. the orbit $\mathrm{Orb}(x,T)$ itself) is dense in H. The Invariant Subset Problem can thus be reformulated as follows: does there exist a minimal action of the semi-group $\mathbb{N}$ on the Hilbert space H?

Of course the Invariant Subspace and Subset Problems make sense on general separable Banach spaces as well, and in this setting both problems admit a negative answer. Enflo [**5**] and Read [**11**] constructed examples of infinite-dimensional separable Banach spaces X and of bounded operators T on X without non-trivial invariant closed subspaces. Read then refined his constructions in several papers such as [**12**], where he gave an example of an operator on ℓ_1 without non-trivial invariant closed subspaces, [**13**] which exhibited an

example of an operator on ℓ_1 without non-trivial invariant closed subsets, and [**14**] which gives examples of operators without non-trivial invariant subspaces on c_0 or on the ℓ_2-direct sum of countably many copies of the James space J. All these counterexamples are constructed on non-reflexive spaces. Both the Invariant Subspace and Subset Problems remain open on reflexive spaces, and on the Hilbert space in particular. We refer the reader to the survey [**1**], the book [**10**] or to the recent book [**2**] for more information. These references are mainly concerned with results in the positive direction, i.e. conditions under which an operator does admit a non-trivial invariant subspace (or subset).

At present, operators on the Hilbert space which seem "closest" to having no non-trivial invariant closed subset are the ones constructed in the paper [**6**]. We call such operators *Read's type operators* because they are constructed by adapting part of the techniques employed by Read in his various constructions to the Hilbert space setting. In order to state the properties of these operators which will be needed in this paper, let us recall that a vector x of H is called *hypercyclic* for an operator T if the orbit $\mathrm{Orb}(x, T)$ is dense in H. We denote by $HC(T)$ the set of such vectors. Then T has no non-trivial invariant closed subset if and only if $HC(T)^c = \{0\}$. The operators constructed in [**6**] have the property that $HC(T)^c$ is "small", in the sense that it is contained in a countable union of closed hyperplanes of H.

Our main result in this paper is the following:

Theorem 1.1. — *There exists a bounded linear operator T acting on a complex separable infinite-dimensional Hilbert space H, and a unitary operator V on H, such that the following property holds true: for every non-zero vector x of H, either x or Vx has a dense orbit under the action of T.*

In other words, $HC(T) \cup HC(V^{-1}TV) = H \setminus \{0\}$. In particular, $\bigcup_{n \geq 0} T^n\{x, Vx\}$ is dense in H for every non-zero vector $x \in H$.

This theorem has the following interesting consequence: denote by $\mathbb{F}_2^+$ the free semigroup with two generators. It can be seen as the set of all finite sequences $(\omega_0, \ldots, \omega_n)$, $n \geq 0$, of zeroes and ones, where the group operation is given by concatenation:

if $\omega = (\omega_0, \ldots, \omega_n)$ and $\theta = (\theta_0, \ldots, \theta_p)$ then $\omega \, . \, \theta = (\omega_0, \ldots, \omega_n, \theta_0, \ldots, \theta_p)$.

If T_0 and T_1 are two bounded linear operators on H, one can define an action ρ of $\mathbb{F}_2^+$ on H in the following way:

$$\rho : \mathbb{F}_2^+ \times H \longrightarrow H$$

$$\big((\omega_0, \ldots, \omega_n), x\big) \longmapsto T_{\omega_0} \ldots T_{\omega_n} x.$$

With the notation above, $\rho(\omega \, . \, \theta, x) = T_{\omega_0} \ldots T_{\omega_n} T_{\theta_0} \ldots T_{\theta_p} x = \rho(\omega, \rho(\theta, x))$. Such an action is called *minimal* if the orbit of every non-zero vector x of H, that is the set $\{\rho(\omega, x), \ \omega \in \mathbb{F}_2^+\}$, is dense in H.

If T and V are the operators given by Theorem 1.1 above, define two operators T_0 and T_1 on H by setting $T_0 = T$ and $T_1 = V^{-1}TV$. Let ρ be the associated action of $\mathbb{F}_2^+$ on H. For every $x \in H \setminus \{0\}$, the orbit $\{\rho(\omega, x), \ \omega \in \mathbb{F}_2^+\}$ contains the sets $\{T_0^n x, \ n \geq 0\}$ and

$\{T_1^n x, \ n \geq 0\}$. Since one of these two sets is dense in H, we obtain in particular that ρ is minimal.

Corollary 1.2. — *There exists a minimal action of the free semi-group with two generators $\mathbb{F}_2^+$ on a complex separable infinite-dimensional Hilbert space H.*

The operator T of Theorem 1.1 is a Read's type operator. Such operators are non-invertible, and this seems to be inherent to the construction. So it is not clear whether it is possible to construct a minimal action of the free group $\mathbb{F}_2$ on a Hilbert space. But if we replace the operators $T_0 = T$ and $T_1 = V^{-1}TV$ above by operators of the form $S_0 = \alpha I + T$ and $S_1 = \alpha I + V^{-1}TV$, where $\alpha \in \mathbb{C}$ is so large that $\alpha I + T$ is invertible, then for every non-zero vector $x \in H$ the linear span of one of the two sets $\{S_0^n x, \ n \geq 0\}$ and $\{S_1^n x, \ n \geq 0\}$ is dense in H. So we obtain in the same way an action ρ of $\mathbb{F}_2$ on H such that for every non-zero vector x of H, the linear span of the orbit $\{\rho(\omega, x), \ \omega \in \mathbb{F}_2\}$ of x under the action of $\mathbb{F}_2$ by ρ is dense in H.

Minimal affine isometric actions of groups on an infinite-dimensional Hilbert space are studied in the paper [**3**], and it is proved here in particular that there exists a minimal isometric action of the free groupe with three generators $\mathbb{F}_3$ on an infinite-dimensional Hilbert space.

The proof of Theorem 1.1 uses some fine properties of the operators constructed in [**6**]. After recalling these briefly in Section 2 we show, using a slightly modified version of the Lomonosov inequality of [**8**], that these Read's type operators on the Hilbert space do have non-trivial invariant closed subspaces, and we give a rather precise description of their set of non-hypercyclic vectors. We prove Theorem 1.1 in Section 3, using this description as well as a construction of specific increasing sequences of subspaces in the vector-valued H^2-space $H^2(\mathbb{D}, \ell^2)$. We also observe there that Theorem 1.1 yields an example of a pair of two unitarily equivalent operators on H which generate $\mathcal{B}(H)$ in the strong (or weak) topology.

2. Main properties of Read's type operators on the Hilbert space

The main properties of the operators constructed in [**6**] which will be of interest to us here are summarized in the following theorem:

Theorem 2.1. — *Let H be a separable infinite-dimensional Hilbert space. There exist bounded operators T on H having the following three properties:*

(P1) *for every vector $x \in H$, the closure of the orbit $Orb(x, T)$ of x under the action of T is a subspace. In other words, the closures of $\{T^n x, \ n \geq 0\}$ and $\mathrm{span}\{T^n x, \ n \geq 0\}$ coincide;*

(P2) *the family $(\overline{Orb}(x, T))_{x \in H}$ of all the closures of the orbits of T is totally ordered by inclusion: for every pair (x, y) of vectors of H, either $\overline{Orb}(x, T) \subseteq \overline{Orb}(y, T)$ or $\overline{Orb}(y, T) \subseteq \overline{Orb}(x, T)$;*

(P') *the operator $T_{|M}$ induced by T on any of its invariant subspaces M is hypercyclic, i.e. there exists a vector $x \in M$ such that $\overline{Orb}(x, T) = M$.*

Property (P') is an easy consequence of properties (P1) and (P2) (see [**6**, Section 5.1]), but we state it explicitly here as it will be needed in the sequel. Another observation which will be important is the following: if $(g_j)_{j \geq 0}$ denotes an orthonormal basis of H, the operators T of [**6**] which are constructed starting from this basis have the form $T = S + K$, where S is a forward weighted shift with respect to the basis $(g_j)_{j \geq 0}$ and K is a nuclear operator: $Sg_j = w_j \, g_{j+1}$ where $0 \leq w_j \leq 2$ for all j and w_j is either 0 or very close to 1 as j tends to infinity, and $\sum_{j \geq 0} ||Kg_j||$ can be made as small as we wish. This kind of property is common to all Read's type operators: in the definition of the vectors Tg_j for $j \geq 0$, the set of all nonnegative integers j is partitioned into two types of intervals, lay-off intervals and working intervals. Working intervals are separated by very long lay-off intervals. When j is not the right endpoint of either a working interval or a lay-off interval, Tg_j is defined as $Tg_j = w_j g_{j+1}$, where the weight w_j is chosen extremely close to 1 when j gets very large. This gives the shift part S of the operator T. When T is a right endpoint of a lay-off or a working interval, $||Tg_j||$ is extremely small and can be chosen to decrease very quickly as j grows. This gives the nuclear part K of the operator, and $T = S + K$. A consequence of this observation is that such operators T do have non-trivial invariant subspaces, although, as will be recalled shortly afterwards, their sets of non-hypercyclic vectors are very small.

Proposition 2.2. — *If T is one of the operators of [**6**] acting on a complex separable infinite-dimensional Hilbert space H, $HC(T)^c$ is a dense linear subspace of H, and T has non-trivial invariant closed subspaces.*

The proof of Proposition 2.2 relies on a refinement of the Lomonosov inequality proven in [**8**]. This inequality motivates the well-known Lomonosov conjecture that any adjoint operator acting on a complex dual separable Banach space X^* has a non-trivial invariant closed subspace. It is stated as follows:

Let X be a complex separable Banach space, and let $\mathcal{A}$ be a weakly closed sub-algebra of $\mathcal{B}(X)$ with $\mathcal{A} \neq \mathcal{B}(X)$. Then there exist two non-zero vectors $x^{**} \in X^{**}$ and $x^* \in X^*$ such that for every $A \in \mathcal{A}$,
$$|\langle x^{**}, A^*x^* \rangle| \leq ||A||_e,$$
where $||A||_e$ denotes the essential norm of A, i.e. the distance of A to the space of compact operators on X.

Here is the slightly stronger statement which will be needed in the present paper. It is completely contained in the proof of [**8**, Th. 1], and so we will only say a few words about it.

Theorem 2.3. — *Let X be a complex separable Banach space, and let $\mathcal{A}$ be a uniformly closed subalgebra of $\mathcal{B}(X)$. Let Q be a closed ball of X^* of positive radius, which does not contain the point 0. Then we have the following alternative: either*

(A1) there exist a positive constant C_Q depending only on Q, a vector $x^* \in Q$ and a non-zero vector $x^{**} \in X^{**}$ such that
$$|\langle x^{**}, A^*x^* \rangle| \leq C_Q \, ||A||_e \quad \text{for every } A \in \mathcal{A}$$

or

(A2) the set $\{A^*x^* \; ; \; A \in \mathcal{A}\}$ is dense in X^* for every vector $x^* \in Q$, and in this case there exists an operator A_0^* on X belonging to $\mathcal{A}$, distinct from the identity operator, and such that $A_0^*x_0^* = x_0^*$ for some vector x_0^* in Q.

Proof. — The proof of Lemma 8 of [**8**] gives the following: if $y^* \in X^*$ is such that $||y^*|| = 3$, and if Q denotes the ball of radius 2 centered at y^*, then either there exist a vector $x^* \in Q$ and a non-zero element x^{**} of X^{**} such that

$$|\langle x^{**}, A^*x^* \rangle| \leq 10\,||A||_e \quad \text{for every } A \in \mathcal{A},$$

or the set $\{A^*x^* \; ; \; A \in \mathcal{A}\}$ is dense in X^* for every vector $x^* \in Q$, and in this case there exists an element A_0 of $\mathcal{A}$ such that $A_0^*x_0^* = x_0^*$ for some vector x_0^* in Q, and 1 is an eigenvalue of A_0^* of finite multiplicity which is an isolated point in the spectrum of A_0^*. Hence (A2) is satified when Q is the closed ball centered at y^* of radius 2

Exactly the same proof shows that if $y^* \in X^*$ is non-zero and Q is the ball of radius r centered at y^*, with $0 < r < ||y^*||$, then either there exist a vector $x^* \in Q$ and a non-zero element x^{**} of X^{**} such that

$$|\langle x^{**}, A^*x^* \rangle| \leq 2(||y^*|| + r)\,||A||_e \quad \text{for every } A \in \mathcal{A},$$

or (A2) holds true. This is the statement of Theorem 2.3. $\qquad\square$

Let us now mention an important consequence of Theorem 2.3, which is a slight generalization of a result of [**7**]:

Corollary 2.4. — *Suppose that T is a bounded operator on a complex separable Hilbert space which is a compact perturbation of a power-bounded operator on H. Then T has a dense set of non-hypercyclic vectors.*

Proof. — By our assumption the operator T can be decomposed as $T = B + K$, where $\sup_{n \geq 0} ||B^n|| \leq M < +\infty$ and K a compact operator on H. Then T^n is a compact perturbation of B^n and $||T^{*n}||_e = ||T^n||_e = ||B^n||_e \leq M$. Let $\mathcal{A}$ be the uniformly closed sub-algebra of $\mathcal{B}(H)$ generated by T^*. Then for every closed ball Q of H with non-empty interior not containing 0, either (A1) or (A2) of Theorem 2.3 above is satisfied. But it is not difficult to see that (A2) can never hold here. Suppose indeed that (A2) is true: then there exist an operator $A_0 \in \mathcal{A}$, not equal to the identity, and a vector $x_0 \in Q$ such that $A_0^*x_0 = x_0$. Since the algebra $\mathcal{A}$ is commutative, the equality $A^*x_0 = A^*A_0^*x_0 = A_0^*A^*x_0$ holds true for every $A \in \mathcal{A}$. But x_0 belongs to Q, so the set $\{A^*x_0 \; ; \; A \in \mathcal{A}\}$ is dense in H by (A2). Hence $A_0 = I$, which is a contradiction. So (A2) cannot be true, and (A1) is satisfied: for every non-empty open ball Q of H not containing 0, there exist a positive constant C_Q and two non-zero vectors $x, y \in H$ with $y \in Q$ such that $\sup_{n \geq 0} |\langle x, T^n y \rangle| \leq C_Q M$. Such vectors y cannot be hypercyclic for T, so T has a dense set of non-hypercyclic vectors. $\qquad\square$

The proof of Proposition 2.2 is now straightforward.

Proof of Proposition 2.2. — If T is one of the operators of [**6**], let $(x_\alpha)_{\alpha \in A}$ be the family of all vectors of H which are non-hypercyclic for T. If for each $\alpha \in A$ we denote by M_α

the subspace $M_\alpha = \overline{\text{Orb}}(x_\alpha, T)$, which is non-trivial, then

$$HC(T)^c = \bigcup_{\alpha \in A} M_\alpha.$$

Since the subspaces M_α are totally ordered by inclusion, the set $HC(T)^c$ is a linear space. It remains to prove that it is dense in H. We have seen that T can be decomposed as $T = S + K$, where the weights w_j of the shift are either 0 or tend to 1 very quickly as j tends to infinity and K is compact. Let S_0 be the weighted shift on H defined by $S_0 g_j = 0$ if $w_j = 0$ and $S_0 g_j = g_{j+1}$ if $w_j > 0$. Then $S - S_0 = L$ is a compact operator. So $T = S_0 + L + K$ is a compact perturbation of S_0, and S_0 is obviously power-bounded since $||S_0|| \leq 1$. Hence T has a dense set of non-hypercyclic vectors by Corollary 2.4. Since all invariant closed subsets of H are automatically subspaces by property (P1) of Theorem 2.1, T has non-trivial invariant closed subspaces. $\square$

Thanks to Proposition 2.2, it is possible to obtain a rather complete description of the structure of the set of non-hypercyclic vectors for these operators.

Proposition 2.5. — *If T is one of the operators of [6] on a complex Hilbert space H, there exists an increasing sequence $(M_n)_{n\geq 0}$ of infinite-dimensional closed subspaces of H such that M_n has infinite codimension as a subspace of M_{n+1} for each $n \geq 0$, $\bigcup_{n\geq 0} M_n$ is dense in H, and $HC(T)^c = \bigcup_{n\geq 0} M_n$.*

Proof. — With the notation used in the proof of Proposition 2.2 above, $HC(T)^c = \bigcup_{\alpha \in A} M_\alpha$ with $M_\alpha = \overline{\text{Orb}}(x_\alpha, T)$. Let $(x_{\alpha_k})_{k\geq 0}$ be a sequence of vectors of $HC(T)^c$ which is dense in $HC(T)^c$. Since $HC(T)^c$ is dense in H, this sequence is dense in H as well. It is proved in [6, Section 5.2] that

$$HC(T)^c = \bigcup_{k\geq 0} M_{\alpha_k}.$$

Since the argument is simple enough, we recall it briefly here: let $x_\alpha \in HC(T)^c$. We wish to show that there exists a $k \geq 0$ such that $M_\alpha \subseteq M_{\alpha_k}$. Suppose that this is not the case. Then $M_{\alpha_k} \subseteq M_\alpha$ for each $k \geq 0$ by property (P2), so that $x_{\alpha_k} \in M_\alpha$ for each $k \geq 0$. Since $\{x_{\alpha_k} \, ; \, k \geq 0\}$ is dense in $HC(T)^c$ and M_α is closed, it follows that the closure of $HC(T)^c$ is contained in M_α, which is an obvious contradiction with Proposition 2.2 and the fact that $M_\alpha \neq H$. Hence $HC(T)^c = \bigcup_{k\geq 0} M_{\alpha_k}$.

Set $M_0 = M_{\alpha_0}$. Let then k_1 be the smallest integer such that $M_0 \subsetneq M_{\alpha_{k_1}}$. Such an integer does exist because the subspaces M_{α_k} are totally ordered by inclusion and it is impossible that $M_{\alpha_k} \subseteq M_0$ for each $k \geq 1$. Set then $M_1 = M_{\alpha_{k_1}}$. We have $M_{\alpha_1} \subseteq M_0, \ldots, M_{\alpha_{k_1-1}} \subseteq M_0$, so that $\bigcup_{k=0}^{k_1} M_{\alpha_k} = M_1$. Continuing in this fashion, we construct by induction a strictly increasing sequence $(k_n)_{n\geq 1}$ of integers such that, setting $M_n = M_{\alpha_{k_n}}$, k_n is the smallest integer such that $M_{n-1} = M_{\alpha_{k_{n-1}}} \subsetneq M_{\alpha_{k_n}} = M_n$. With this definition we have as above $\bigcup_{k=0}^{k_n} M_{\alpha_k} = M_n$, $M_n \subsetneq M_{n+1}$, and $\bigcup_{k\geq 0} M_{\alpha_k} = \bigcup_{n\geq 0} M_n = HC(T)^c$.

Each subspace M_n of H is infinite-dimensional by property (P') and the fact that no operator on a finite-dimensional space admits a hypercyclic vector. It remains to prove that

for each $n \geq 0$, M_n is a subspace of M_{n+1} of infinite codimension in M_{n+1}: the operator $T_{|M_{n+1}}$ induced by T on M_{n+1} admits a hypercyclic vector by property (P'), so the quotient operator $\overline{T}_{|M_{n+1}}$ on the non-zero quotient space M_{n+1}/M_n admits a hypercyclic vector as well. So M_{n+1}/M_n is necessarily infinite-dimensional, and this proves our claim. $\qquad\square$

As a consequence of Proposition 2.5 we obtain:

Proposition 2.6. — *Let $(E_n)_{n \geq 0}$ be a sequence of closed subspaces of the Hilbert space H having the following properties:*

(1) $E_n \subseteq E_{n+1}$ for each $n \geq 0$,

(2) each subspace E_n is infinite-dimensional and has infinite codimension as a subspace of E_{n+1},

(3) the union $\bigcup_{n \geq 0} E_n$ is dense in H.

*Let T be one of the operators of [**6**], having the property given by Proposition 2.5. There exists a unitary operator U on H such that*

$$HC(UTU^{-1})^c = \bigcup_{n \geq 0} E_n.$$

Proof. — Let T be one of these operators, and let $(M_n)_{n \geq 0}$ be the increasing sequence of subspaces such that $HC(T)^c = \bigcup_{n \geq 0} M_n$ given by Proposition 2.5. Let $(g_{k,0})_{k \geq 0}$ be an orthonormal basis of M_0, and for each $n \geq 1$ let $(g_{k,n})_{k \geq 0}$ be an orthonormal basis of the orthogonal complement $M_n \ominus M_{n-1}$ of M_{n-1} in M_n. If $(E_n)_{n \geq 0}$ is a sequence of subspaces having the properties stated in Proposition 2.6, let $(e_{k,0})_{k \geq 0}$ be an orthonormal basis of E_0 and $(e_{k,n})_{k \geq 0}$ an orthonormal basis of $E_n \ominus E_{n-1}$ for each $n \geq 1$. As each one of the two sets $\{g_{k,n} \, ; \, n \geq 0, \ k \geq 0\}$ and $\{e_{k,n} \, ; \, n \geq 0, \ k \geq 0\}$ span a dense subspace of H, each of them forms an orthonormal basis of H, and there exists a unitary operator U on H such that $Ug_{k,n} = e_{k,n}$ for each $n \geq 0$ and $k \geq 0$. It follows that $U(M_n) = E_n$ for each $n \geq 0$, and thus the operator UTU^{-1} satisfies $HC(UTU^{-1})^c = \bigcup_{n \geq 0} E_n$. $\qquad\square$

We are now ready for the proof of Theorem 1.1.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 relies on the following simple idea: let $(E_n)_{n \geq 0}$ and $(F_n)_{n \geq 0}$ be two sequences of closed subspaces of H satisfying properties (1), (2), and (3) of Proposition 2.6. Let $T \in \mathcal{B}(H)$ be one of the operators of [**6**], and let U_E and U_F be the two associated unitaries given by Proposition 2.6:

$$HC(U_E T U_E^{-1})^c = \bigcup_{n \geq 0} E_n \quad \text{and} \quad HC(U_F T U_F^{-1})^c = \bigcup_{n \geq 0} F_n.$$

If we manage to construct the sequences of subspaces $(E_n)_{n \geq 0}$ and $(F_n)_{n \geq 0}$ in such a way that

$$\left(\bigcup_{n \geq 0} E_n \right) \cap \left(\bigcup_{n \geq 0} F_n \right) = \{0\},$$

then we will get that $HC(U_E T U_E^{-1})^c \cap HC(U_F T U_F^{-1})^c = \{0\}$. This means that

$$HC(T)^c \cap HC(V^{-1}TV)^c = \{0\},$$

where $V = U_F^{-1} U_E$, i.e. that $HC(T) \cup HC(V^{-1}TV) = H \setminus \{0\}$, and this is exactly the statement of Theorem 1.1. So we see that everything boils down to showing the following lemma:

Lemma 3.1. — *Let H be a complex separable infinite-dimensional Hilbert space. There exist two sequences $(E_n)_{n \geq 0}$ and $(F_n)_{n \geq 0}$ of closed subspaces of H which both satisfy assertions (1), (2), and (3) of Proposition 2.6 and are such that*

$$\Big(\bigcup_{n \geq 0} E_n \Big) \cap \Big(\bigcup_{n \geq 0} F_n \Big) = \{0\}.$$

Proof. — Let us first work in the Hardy space $H^2(\mathbb{D})$, and consider for $n \geq 0$ the spaces $G_n = \mathrm{span}[1, z, \ldots, z^n]$ of polynomials of degree at most n. Of course, $G_n \subseteq G_{n+1}$ and $\dim(G_{n+1} \ominus G_n) = 1$ for each $n \geq 0$, and $\bigcup_{n \geq 0} G_n$ is dense in $H^2(\mathbb{D})$. Let now $(z_j)_{j \geq 0}$ be the sequence of points of $\mathbb{D}$ defined by $z_j = 1 - 1/(j+1)^2$ for $j \geq 0$. For each $n \geq 0$, consider the Blaschke product B_n given by

$$B_n(z) = \prod_{j \geq n} \frac{z_j - z}{1 - z_j z} \quad (z \in \mathbb{D})$$

and the subspaces $K_n = B_n H^2(\mathbb{D})$ of $H^2(\mathbb{D})$. This is an increasing sequence of infinite-dimensional closed subspaces of $H^2(\mathbb{D})$, and $\dim(K_{n+1} \ominus K_n) = 1$ for each $n \geq 0$. Let us now show that the increasing union $\bigcup_{n \geq 0} K_n$ is dense in $H^2(\mathbb{D})$. Let $f \in H^2(\mathbb{D})$ be a function which is orthogonal to all subspaces K_n. Denote by $L^2_-(\mathbb{T})$ the set of functions g in $L^2(\mathbb{T})$ such that $\widehat{f}(j) = 0$ for each $j \geq 0$. Saying that f belongs to $H^2(\mathbb{D}) \ominus B_n H^2(\mathbb{D})$ means that $f \in H^2(\mathbb{D}) \cap B_n L^2_-(\mathbb{T})$. So there exists for each $n \geq 0$ a function $g_n \in L^2_-(\mathbb{T})$ such that $f = B_n g_n$. This means that $f(e^{i\theta}) = B_n(e^{i\theta}) g_n(e^{i\theta})$ for almost every $e^{i\theta} \in \mathbb{T}$, and hence there exists a subset Ω of $\mathbb{T}$ of measure 1 such that for every $e^{i\theta} \in \Omega$ and every $n \geq 0$, $f(e^{i\theta}) = B_n(e^{i\theta}) g_n(e^{i\theta})$. Now the estimates

$$|1 - B_n(e^{i\theta})| \leq \sum_{j \geq n} \left| \frac{z_j - e^{i\theta}}{1 - z_j e^{i\theta}} - 1 \right| \leq \sum_{j \geq n} (1 - z_j) \frac{2}{|1 - z_j e^{i\theta}|}$$

$$\leq \frac{2}{|1 - e^{i\theta}| - \frac{1}{(n+1)^2}} \sum_{j \geq n} \frac{1}{(j+1)^2}, \quad e^{i\theta} \in \mathbb{T} \setminus \{1\}$$

show that whenever $e^{i\theta} \in \mathbb{T} \setminus \{1\}$, $B_n(e^{i\theta})$ tends to 1 as $n \to +\infty$. It follows that for every $e^{i\theta} \in \Omega \setminus \{1\}$, $g_n(e^{i\theta})$ tends to $f(e^{i\theta})$ as $n \to +\infty$. Since $|g_n(e^{i\theta})| = |f(e^{i\theta})|$ almost everywhere on $\mathbb{T}$ and $g_n \in L^2_-(\mathbb{T})$ for each $n \geq 1$, by Lebesgue dominated convergence theorem we obtain that $f \in L^2_-(\mathbb{T})$. But f belongs to $H^2(\mathbb{D})$ as well, and so $f = 0$. We have thus proved that $\bigcup_{n \geq 0} K_n$ is dense in $H^2(\mathbb{D})$.

The next step of the proof is to show that

$$\Big(\bigcup_{n \geq 0} G_n \Big) \cap \Big(\bigcup_{n \geq 0} K_n \Big) = \{0\}.$$

This is straightforward: if f belongs to one of the spaces $K_n = B_n H^2(\mathbb{D})$, f vanishes at all the points z_j for $j \geq n$, and f has infinitely many zeroes. Hence f cannot be a polynomial unless it vanishes identically.

Consider now the space $H = \bigoplus_{\ell_2} H^2(\mathbb{D})$ which is the ℓ_2-sum of countably many copies of $H^2(\mathbb{D})$. This space can also be seen as the vector-valued H^2-space $H^2(\mathbb{D}, \ell_2)$. Set, for each $n \geq 0$, $E_n = \bigoplus_{\ell_2} G_n$ and $F_n = \bigoplus_{\ell_2} K_n$. Obviously E_n and F_n are infinite-dimensional, $E_n \subseteq E_{n+1}$ and $F_n \subseteq F_{n+1}$ for each $n \geq 0$. Since $\dim(G_{n+1} \ominus G_n) = \dim(K_{n+1} \ominus K_n) = 1$, it is clear that E_n and F_n are of infinite codimension in E_{n+1} and F_{n+1} respectively. Lastly, since $\bigcup_{n \geq 0} G_n$ and $\bigcup_{n \geq 0} K_n$ are both dense in $H^2(\mathbb{D})$, $\bigcup_{n \geq 0} E_n$ and $\bigcup_{n \geq 0} F_n$ are dense in H. So (1), (2), and (3) of Proposition 2.6 are satisfied. It remains to see that $E_n \cap F_m = \{0\}$ for each $m, n \geq 0$, but this follows directly from the fact that $G_n \cap K_m = \{0\}$. This finishes the proof of Lemma 3.1. $\qquad\square$

Theorem 1.1 is proved.

It is interesting to note the following consequence of Theorem 1.1 and the Lomonosov inequality:

Proposition 3.2. — *The two operators T and $V^{-1}TV$ given by Theorem 1.1 are unitarily equivalent, and the weakly (strongly) closed algebra $\mathcal{A}$ they generate is equal to $\mathcal{B}(H)$. The uniformly closed algebra $\mathcal{R}$ they generate contains all compact operators on H.*

Proof. — First recall that the closures of a linear subspace of $\mathcal{B}(H)$ for the weak and strong topologies always coincide, so that we can indifferently consider $\mathcal{A}$ as the weak or strong closure of the algebra generated by T and $V^{-1}TV$. Suppose that $\mathcal{A} \neq \mathcal{B}(H)$. Then we can apply the Lomonosov inequality, and obtain two non-zero vectors x and y of H such that $|\langle x, Ay \rangle| \leq \|A\|_e$ for every $A \in \mathcal{A}$. So in particular we get that for every $n \geq 0$, $|\langle x, T^n y \rangle| \leq \|T^n\|_e = \|S_0^n\|_e \leq 1$, and $|\langle x, V^{-1}T^n V y \rangle| \leq \|V^{-1}T^n V\|_e = \|V^{-1}S_0^n V\|_e \leq 1$. This contradicts the fact, given by Theorem 1.1, that either y or Vy has a dense orbit under the action of T. The statement about $\mathcal{R}$ is proved in exactly the same way, using [**8**, Th. 3] which states that either there exist two non-zero vectors x and y of H such that $|\langle x, Ry \rangle| \leq \|R\|_e$ for every $R \in \mathcal{R}$, or $\mathcal{R}$ contains all compact operators on H. $\qquad\square$

There are of course much simpler examples of pairs (R, S) of operators on H which generate $\mathcal{B}(H)$: as was pointed out to us by Lyudmila Turowska, it suffices to take for S the forward shift with respect to an orthonormal basis $(g_j)_{j \geq 0}$ of H, and for R the backward shift with respect to this same basis. Then $RS - SR$ is the rank one operator $\langle g_0, \cdot \rangle g_0$, and from this one easily sees that the algebra generated by R and S is weakly (strongly) dense in $\mathcal{B}(H)$. This is a particular case of a result of [**9**], where it is shown that for any separable real or complex Banach space X, $\mathcal{B}(X)$ can be generated by two operators R and S. Finding a minimal number of operators belonging to a certain subclass of $\mathcal{B}(H)$ and generating $\mathcal{B}(H)$ is a problem which has been much studied. We refer the reader for instance to the paper [**4**], where it is proved that $\mathcal{B}(H)$ can be generated by two unitary operators, and by no less than three projections, and to [**9**] and the references therein.

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SOPHIE GRIVAUX, CNRS, Laboratoire Paul Painlevé, UMR 8524, Université Lille 1, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex, France ● *E-mail :* `grivaux@math.univ-lille1.fr`

MARIA ROGINSKAYA, Department of Mathematical Sciences, Chalmers University of Technology, SE-41296 Göteborg, Sweden, *and* Department of Mathematical Sciences, Göteborg University, SE-41296 Göteborg, Sweden ● *E-mail :* `maria@chalmers.se`